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Optimal Control of Stochastic Hybrid Systems Based on Locally Consistent Markov Decision Processes

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Abstract

This paper applies an approach for approximating controlled stochastic diffusion to hybrid systems. Stochastic hybrid systems are approximated by locally consistent Markov decision processes that preserve local mean and covariance. A randomized switching policy is introduced for approximating the dynamics on the switching boundaries. The validity of the approximation is shown by solving the optimal control problem of minimizing a cost until a target set is reached using dynamic programming. It is shown that using the randomized switching policy, the solution obtained based on the discrete approximation converges to the solution of the original problem.

1 Introduction

Many practical systems such as automobiles, chemical processes, and autonomous vehicles are best described by dynamics that comprise continuous state evolution within a mode of operation and discrete transitions from one mode to another, either controlled or autonomous. Such systems often interact with the environment in the presence of uncertainty and variability. Stochastic hybrid systems can model complex dynamics, uncertainty, and multiple modes of operations and they can support high-level control specifications that are required for design of autonomous or semi-autonomous applications.

Our goal in this work is to develop a systematic way to approximate stochastic hybrid systems that is amenable to computational methods. We extend the approach presented in [19] to hybrid systems. The basic idea of the approach is to approximate the original processes by appropriate Markov Decision Processes defined on a discrete state space. The approximation is achieved by

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constructing locally consistent MDPs that preserve local mean and covariance. Based on the discrete approximation, the stochastic optimal control problem is solved using dynamic programming. Given the value function for the discrete approximation, a control law for the actual system can be computed on-line using multi-linear interpolation. The main advantage of the approach is that the control based on the discrete approximation is directly related to the original processes through the notion of local consistency and further, it is shown that the solution converges to the solution of the original problem.

Although the approach has been already applied to several classes of stochastic systems [19], to our knowledge, the application to stochastic hybrid systems is novel. The extension of the approach to hybrid systems faces the significant challenge of approximating the dynamics in the neighborhood of the switching boundaries. The main contribution of this paper is the introduction of a randomized switching policy that guarantees under appropriate conditions continuity of the switching times. Based on this idea, convergence of the approximating processes to the stochastic hybrid process can be shown using a straightforward extension of the techniques presented in [19]. It should be noted that the approach in [19] can handle discontinuous dynamics and discontinuous cost functions. However, it has been recognized that the approximation method in the neighborhood of the discontinuity sets is a significant issue since it affects the robustness of the solution. Our proposed method first transforms the partition of the state space to a cover. Then, the discrete process that approximates the hybrid system is defined using random switching times. The method of transforming the state space partition to a cover has been used also for removing Zeno behavior from hybrid system models as well as improving the robustness of estimation algorithms in the presence of process and measurement noise [18]. The advantages with respect to robustness stem from the continuity of analog-to-digital maps based on covers of the state space [21].

Several modeling paradigms for stochastic hybrid systems have been already proposed. A stochastic hybrid system scheme that allows the continuous flows at each discrete location to be characterized by stochastic differential equations is described in [16]. An extension of this model that satisfies a Markov property is presented in [6] and a method to study the reachability problem is proposed. A similar model based on piecewise deterministic Markov processes is presented in [5] for studying a probabilistic reachability problem. Probabilistic hybrid automata are introduced in [15] for estimation and fault diagnosis. Communicating piecewise Markov processes are proposed as compositional specifications for stochastic hybrid systems in [23] with an emphasis on modeling concurrency. Applications of stochastic hybrid systems to air traffic management systems are presented in [22, 10]. A stochastic hybrid system with application to communication networks is presented in [14]. A modeling framework and a simulation environment for concurrent stochastic hybrid systems is presented in [3]. The differences between these models are in the way randomness affects the continuous and discrete dynamics and their interaction.

In this paper, we consider a model similar to that given in [16] but we assume that the stochastic differential equations that describe the continuous dynamics are controlled diffusions. The convergence results are shown in the space of piecewise continuous functions that are continuous from the right and have limits from the left. To simplify the notation, we also assume that the dispersion matrix is independent of the discrete state and control. Finally, we consider only autonomous switchings. Controlled switchings can be easily incorporated in the computational methods since they are based on discrete approximations and in the convergence results are based on a relaxed control representation.

To investigate the validity of the approximation we study the optimal control problem of minimizing a cost until a target set is reached. Optimal control of hybrid systems has attracted considerable attention in computer science and control engineering. There are both theoretical results and computational methods developed for non-stochastic hybrid systems. Optimal control problems based on a unified hybrid model have been formulated in [4]. The main result in this work is the derivation of generalized quasi-variational inequalities that characterize the optimal solutions. Several approaches including discretization techniques have been proposed to solve these inequalities, however, efficient computational methods have not been developed. An approach for control of hybrid systems based on calculus of variations that employs chattering approximations to optimal control solutions has been proposed in [9]. Several other design approaches based on optimal control have been proposed [26, 7]. These approaches are based on a hierarchical structure obtained by imposing simplifying assumptions on the system (for example, order of continuous, discrete optimization) which often can be restrictive. A formulation of the maximum principle for hybrid systems has appeared in [24]. There are additional approaches to optimal control [1] as well as efforts to extend existing results to stochastic optimal control methods [2] but computational methods have not been proposed.

Sufficient and necessary conditions for the stochastic optimal control problem of switching diffusions have been presented in [11]. These conditions require the solution of a partial differential equation (PDE) that cannot be solved analytically but only in cases where the coupling between the continuous and discrete dynamics is very simple. Controlled switching diffusions have been used to model hybrid processes in [12] and dynamic programming equations have been derived for the infinite horizon stochastic optimal control problem. These are coupled elliptic equations that in the general case can be solved numerically using discretization methods. A dynamic programming method based on discretization has been also proposed in [13]. Our approach is also based on discretization but it provides a significant advantage. The solution based on the approximating model is directly related to the solution of the original problem through the notion of local consistency and it converges as the discretization becomes finer.

The main research challenge that arises is the scalability of the proposed computational methods. The state space of the approximating MDP increases exponentially with the dimension of the state space. This limits the application of the approach to low-dimensional systems. Development of efficient computational methods for analysis and design based on the locally consistent MDPs is currently under investigation.

The paper is organized as follows. Section 2 presents the stochastic hybrid system model. The stochastic optimal control problem formulation is presented in Section 3. The approximating method is described in Section 4. The discretized optimal control problem is presented in Section 5. Section 6 contains the convergence results. Finally, the approach is illustrated in Section 7 with a simplified 3-dimensional example of a car with two gears.

2 Stochastic Hybrid Systems

Definition 1 A stochastic hybrid system (SHS) is defined as $(X, Q, U, \Omega, A, f, \sigma, \delta, R, (x_0, q_0))$ where

- $X \subseteq \mathbb{R}^d$ is the continuous state space,
- $Q, |Q| = N$ is a finite set of discrete states,
- $U = \{U_q\}_{q \in Q}, U_q \subset \mathbb{R}^{m_q}$ is a collection of continuous control input sets,
- $\Omega = \{\Omega_q\}_{q \in Q}, \Omega_q \subset \mathbb{R}^d$ is a partition of X,
- $A = \{A_q\}_{q \in Q}, A_q \subseteq \partial\Omega_q$ is a collection of autonomous switching sets,
- $f : X \times Q \times U_q \to X$ and $\sigma : X \to \mathbb{R}^{d \times p}$ are the controlled drift vectors and dispersion matrices respectively,
- $\delta : Q \times A \to Q$ is the autonomous switching map,
- $R : Q \times A \to \mathcal{P}(X)$[1] is a reset map which assigns to each q and $x \in A_q$ a reset probability kernel on X concentrated on $\Omega_{q'}$ where $q' = \delta(q, x)$,
- (x_0, q_0) is an initial probability measure on $X \times Q$.

To define the *execution* of the SHS, we consider an $\mathbb{R}^p$-valued Wiener process (Brownian motion) $w(t)$ and a sequence of stopping times $\{t_0 = 0, t_1, t_2, \ldots, \}$ that represent the times when the continuous and discrete dynamics interact. Let the state at time t_i be $(x_i, q_i) = (x(t_i), q(t_i))$ with $x_i \in \Omega^0_{q_i}$ [2]. While the continuous state stays in $\Omega^0_{q_i}$, $x(t)$ is evolving according to the stochastic differential equation (SDE)

$$dx = f(x, q, u)dt + \sigma(x)dw \tag{1}$$

[1] $\mathcal{P}(X)$ denotes the family of probability measures on X.

[2] Ω^0 denotes the interior of the set Ω

where the discrete state $q(t) = q_i$ remains constant and the solution is understood using the Itô stochastic integral [20]. Let $t_{i+1} = \inf\{t > t_i : x(t) \notin \Omega^0_{q_i}\}$. At t_{i+1} an autonomous discrete transition and a reset of the continuous state occur. The new discrete state is $q_{i+1} = \delta(q_i, x(t^-_{i+1}))$. The new continuous state $x(t_{i+1})$ is selected randomly according to the probability measure $R(q_i, x(t^-_{i+1}))(\Xi)$ where $\Xi \subset \Omega_{q_i+1}$ is a measurable set. The evolution of $x(t)$ is then described by the SDE (1) with $q(t) = q_{i+1}$ and initial condition $x(t_{i+1})$ until the next switching time.

It is assumed that the functions $f(x, q, u)$ and $\sigma(x)$ are Lipschitz continuous in x, then the SDE (1) has a unique solution. We also assume that every point $x \in A_q$ is a *regular* point for the autonomous switching set A_q. Note that if $x \in A_q$ is regular for A_q, then a sample path of (1) which starts at x will not remain in A_q for a nonempty time interval [17]. If x is regular for A_q then it is also regular for a neighborhood of A_q around x and we can conclude that

$$\lim_{\epsilon \to 0} \int_0^t P[x(s) \in N_\epsilon(A_q)] ds = 0$$

where $N_\epsilon(A_q) = \{x : d(x, A_q) \le \epsilon\}$ and $d(x, A_q)$ is the Euclidean distance between x and A_q. The regularity assumption ensures that the sample paths would transverse the autonomous switching sets and therefore, we can consider that the autonomous switchings occur instantaneously. A sufficient condition for the regularity assumption is that the set A_q has dimension $d - 1$ and the diffusion $a(x) = \sigma(x)\sigma^T(x)$ is non-degenerate. If $a(x)$ is degenerate, it is possible to satisfy the regularity assumptions by assuming that the drift term f does not vanish in the switching boundary. Let $x(t_{i+1})$ be the continuous state after a discrete transition. We also assume that for every $x(t_{i+1}) \in \Xi$, $d(x(t_{i+1}, A) \ge \epsilon > 0$ and $\exists \delta > 0$ such that $P(\inf\{t > t_{i+1}, x(t) \in A\} \ge \delta) = 1$ and therefore, $t_{i+1} - t_i > \delta, i = 1, 2, \ldots$, with probability 1. Finally, the continuous control is a measurable stochastic process $u(t)$ taking values in a compact set. The control policy $u(t)$ is said to be *admissible* if (i) it is non-anticipative with respect to the Wiener process $w(t)$, (i.e. $u(t)$ is independent of $w(s) - w(t)$, $\forall s > t$).

3 Stochastic Optimal Control

In this section, we describe the problem of minimizing a cost until a target set is reached. This problem is used to demonstrate the validity of the discrete approximations proposed in the paper. Figure 1 illustrates the optimal control problem. The target set $G \subset \mathbb{R}^d$ is assumed to be a compact set with a smooth boundary ∂G which satisfies the same regularity conditions as the autonomous switching sets. Further, we assume that $x_0 \notin G$ and $G \subset \Omega_q$ for some $q \in Q$. We define the *stopping time* τ by $\tau = \inf\{t : x(t) \in \partial G\}$. If the stopping time is not defined then the value of τ is set to infinity.

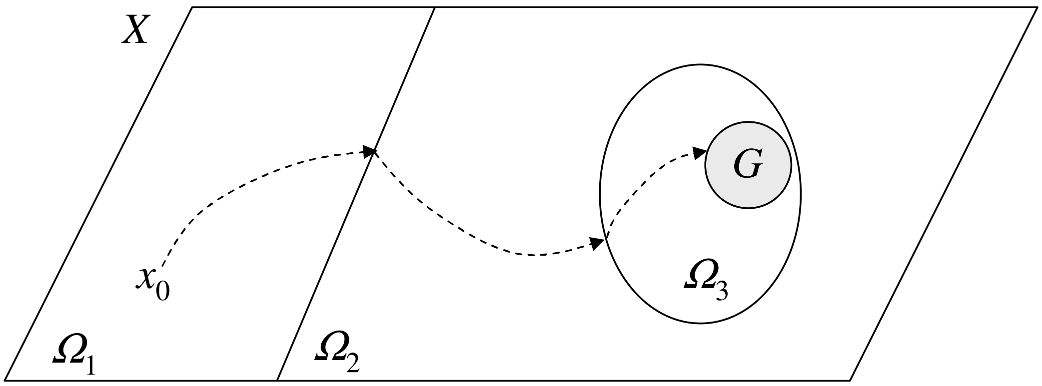

Figure 1: Optimal control problem

Given a stochastic hybrid system, a target set G, an initial state (x_0, q_0) at $t_0 = 0$, and a discount factor $\beta \geq 0$ the optimal control problem is formulated as the minimization of the cost

$$W(x_0, q_0, u) = E\left[\int_0^\tau e^{-\beta s} k(x(s), q(s), u(s))ds + e^{-\beta\tau} g(x(\tau))\right] \tag{2}$$

with respect to the admissible controls $u(t)$.

Next, we describe an approach for stochastic optimal control based on dynamic programming. To ensure that the stopping time and the cost (2) are well-defined and bounded, we assume that if $\beta = 0$ then for every initial state (x_0, q_0) there exists an admissible control policy so that the state will reach the target set G. If $\beta > 0$ the cost will be bounded even if the stopping time is not.

The value function is defined by

$$V(x_0, q_0) = \inf_u W(x_0, q_0, u), x_0 \in \Omega_{q_0}.$$

Based on a standard dynamic programming argument, we can formulate the following result. Since the initial condition can be arbitrary we will denote the value function by $V(x, q)$.

Theorem 1 *Given a SHS and the cost (2), an optimal admissible control policy $u(x)$ must satisfy the conditions*

$$\inf_u \left[\nabla V(x,q) f(x,q,u) + \frac{1}{2} tr(\nabla^2 V(x,q) a(x))\right] = 0,$$

$$\forall q \in Q, \forall x \in \Omega_q^0$$

$$V(x', q') \leq V(x, q),\ \forall q \in Q, x \in A_q, q' = \delta(q, x), x' \sim R(q, x)(\Xi)$$

$$V(x, q) = g(x), \forall x \in \partial G, q \in Q : G \subset \Omega_q.$$

In addition, the following verification theorem can be proved in a straightforward manner.

Theorem 2 *Suppose that there exist $V(x,q)$ twice differentiable in x, and bounded in Ω_q^0 and a feedback control $\bar{u}(x)$ such that the conditions of Theorem 1 hold and $W(x,q,\bar{u})$ is bounded. Then $V(x,q)$ is the optimal cost and $\bar{u}(x)$ the optimal control.*

In practice, computing the optimal value function $V(x,q)$ can be very difficult and usually requires computational methods based on discretization of the state space. In this paper, we employ a discretization method for the approximation of stochastic hybrid systems by appropriately chosen MDPs [19]. The SDE at every location q of the hybrid system is approximated by a controlled Markov process that evolves in a state space that is a discretization of the region Ω_q. The criterion which must be satisfied by the approximating MDP is *local consistency*. Local consistency means that the conditional mean and covariance of the MDP are proportional to the local mean and covariance of the original process. An approximation parameter h analogous to a "finite element size" parameterizes the approximating Markov process. As h goes to zero, the local properties of the MDP resemble the local properties of the original stochastic process. Application of the approach to stochastic hybrid systems requires approximating the behavior at the switching boundaries. Suitable approximations must ensure convergence of the approximating processes to the original stochastic hybrid system.

4 Locally Consistent Markov Decision Processes

This section presents the locally consistent MDP that will be used to approximate the SHS. This work employs the approximation method presented in [19] for computing locally consistent MDPs. Although this method can be used to approximate the continuous dynamics, it cannot be applied for approximating the behavior at the switching boundaries. In this section, first we discuss some necessary background for the approximation method of [19] that will be extended for approximating SHS and then we focus on the approximation at the switching boundaries.

4.1 Background Material

Consider the SDE (1) evolving in Ω_q^0. The local mean and covariance on the interval $[0,\delta]$ are

$$E[x(\delta)-x] = f(x(t),q(t),u(t))\delta + o(\delta)$$

$$E[(x(\delta)-x)(x(\delta)-x)^T] = \sigma(x(t))\sigma^T(x(t))\delta + o(\delta).$$

Let $\{\xi_n^h\}$ be an MDP on a discrete state space $S_q^h \subset \Omega_q$ with transition probabilities denoted by $p((x,q),(y,q)|u)$. A locally consistent MDP must satisfy

$$E[\Delta\xi_n^h] = f(x,q,u)\Delta t^h(x,q,u) + o(\Delta t^h(x,q,u))$$

$$E[(\Delta\xi_n^h - E[\Delta\xi_n^h])(\Delta\xi_n^h - E[\Delta\xi_n^h])^T] = \sigma(x)\sigma^T(x)\Delta t^h(x,q,u) + o(\Delta t^h(x,q,u))$$

where $\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h, \xi_n^h = x$ and $\Delta t^h(x,q,u)$ are appropriate interpolation intervals (or the "holding times") for the MDP. In general, the control action u can take values in a compact set in some topological space. We say that a control policy $\{u_n^h, n < \infty\}$ is *admissible* if the chain has the Markov property under this policy.

The transition probabilities $p((x,q),(y,q)|u)$ and the interpolation intervals can be computed systematically from the parameters of the SDE (details can be found in [19]). In the case the diffusion matrix $a(x) = \sigma(x)\sigma^T(x)$ is diagonal and for a regular grid where e_i is unit vector in the i^{th} direction, the transition probabilities are

$$p((x,q),(x \pm he_i,q)|u) = \frac{a_{ii}(x)/2 + hf_i^{\pm}(x,q,u)}{Q^h(x,q,u)} \tag{3}$$

and the interpolation interval is

$$\Delta t^h(x,q,u) = \frac{h^2}{Q^h(x,q,u)} \tag{4}$$

where $Q^h(x,q,u) = \sum_i [a_{ii}(x) + h|f_i(x,q,u)|]$ and $a^+ = \max\{a,0\}$ and $a^- = \max\{-a,0\}$ denote the positive and negative parts of a real number.

It should be noted that an MDP that is locally consistent with the SDE (1) is not unique. Any reasonable approximation that satisfies the local consistency conditions can be used. Optimization algorithms for MDPs employ iteration in policy/value space. To perform efficiently the minimization over the admissible controls at every iteration (see Section 5) it is desirable to eliminate the control dependence u in the denominators of the transition probabilities and the interpolation interval. This is always possible if the SDE (1) is affine in the controls [19] and can be accomplished by defining $\bar{Q}^h(x,q) = \max_{u \in U_q} Q^h(x,q,u)$ and replace $Q^h(x,q,u)$ by $\bar{Q}^h(x,q)$ in equations (3) and (4).To ensure that the transition probabilities sum to one for each x and u, we introduce

$$p((x,q),(x,q)|u) = 1 - \sum_{y, y \neq x, q', q' \neq q} p((x,q),(y,q')|u).$$

It can be shown that the difference between the old and the new values of the transition probabilities is $O(h)$ and therefore, the new transition probabilities and interpolation interval are also locally consistent with (1) [19].

4.2 Switching Boundaries

The method discussed above approximates the continuous dynamics by discrete MDPs only in the interior of the regions Ω_q. Approximating stochastic hybrid systems requires defining the MDP in the neighborhood of the switching boundaries in a way that preserves local consistency. There are two particular forms of switching boundaries of interest, smooth hypersurfaces and boundaries of polyhedral sets that have "corners". Here, we consider the case of smooth hypersurfaces. The method can be extended for the case of switching boundaries with "corners" in a straightforward manner and details are omitted due to length limitations. The main idea in this paper consists of the following steps: (i) transform the partition of the SHS to a cover, (ii) define appropriate random switching functions for approximating the behavior at the boundaries.

First, the partition $\Omega = \{\Omega_q\}$ is transformed to a cover. Consider the region Ω_q and denote its boundary A_q as

$$A_q = \{x \in \mathbb{R}^d : a_q(x) = 0\}$$

where $a_q : \mathbb{R}^d \to \mathbb{R}$ is assumed to be a smooth functional. The functional a_q must satisfy the condition $\nabla a_q(x) \neq 0, \ \forall x \in A_q$ which ensures that the boundary is an $(d-1)$-dimensional hypersurface separating the state space. Assume without loss of generality that $\forall x \in \Omega_q^0$ we have $a_q(x) < 0$. The region Ω_q is expanded to Ω'_q defined by

$$\Omega'_q = \{x \in \mathbb{R}^d : a_q(x) - \gamma(h) = 0\}$$

where $\gamma(h) > h > 0$ for every $h > 0$ and $\gamma(h) \to 0$ as $h \to 0$. By expanding Ω_q to Ω'_q we obtain $\Omega' = \{\Omega'_q\}, q \in Q$. Since $\bigcup_q \Omega_q \subseteq \bigcup_q \Omega'_q$ for every q, Ω' is a cover of the state space X of the SHS.

Let $\{(\xi_n^h, q_n^h)\}$ be an MDP on a discrete state space $S = \{(x,q) \in S^h \times Q : x \in \Omega'_q\}$ with transition probabilities denoted by $\tilde{p}\left((x,q),(x',q')\right)|u)$. For all states (x,q) such that $a_q(x) < 0$ (interior of Ω_q), the system cannot switch and the transition probabilities are computed so that the MDP is locally consistent with the corresponding SDE. Hence, $\forall (x,q) \in S : a_q(x) < 0$ we have

$$\begin{aligned} \tilde{p}\left((x,q),(x \pm he_i, q)|u\right) &= p((x,q),(x \pm he_i, q)|u), \\ \tilde{p}\left((x,q),(x,q)|u\right) &= p((x,q),(x,q)|u) \end{aligned}$$

and

$$\tilde{p}\left((x,q),(x,q')|u\right) = 0, \text{ if } q \neq q'.$$

The switching behavior of the SHS is approximated by introducing random switching times and discretizing the reset maps. For each boundary $A'_q = \{x \in \mathbb{R}^d : a_q(x) = \gamma(h)\}$, we define the *switching rate function* $\lambda_q(x)$ such that $\lambda_q(x)$ is continuous on $O_q = \{x \in \mathbb{R}^d : 0 \leq a_q(x) \leq \gamma(h)\}$, $\lambda_q(x) = 0$ if $a_q(x) = 0$, and $\lambda_q(x) \to \infty$ as $x \to A'_q$. We also approximate the reset map $R(q,x)(\Xi)$ by a discrete transition probability kernel. If a discrete transitions $q \to q'$ occurs, the next continuous state is selected randomly from the grid points that belong to Ξ according to a uniform distribution. Let $x_i \in \Xi, i = 1, 2, \ldots, \zeta$, then

$$p((x,q),(x',q')|u) = \begin{cases} 1/\zeta & \text{if } x' \in \Xi \\ 0 & \text{otherwise} \end{cases} .$$

Consider the interpolation intervals $\Delta\tau_n^h = \Delta t^h(\xi_n^h, q_n^h, u_n^h)$, we define the process $\{q_n^h\}$

$$P[q_{n+1}^h \text{ changes in } \Delta\tau_n^h | q_n^h, \xi_n^h, u_n^h] = \begin{cases} 1, & \text{if } \xi_n^h \in A'_{q_n^h} \\ 1 - e^{-\lambda_{q_n^h}(\xi_n^h)\Delta t^h(\xi_n^h, q_n^h, u_n^h)}, & \text{if } \xi_n^h \in N_h(A'_{q_n^h}) \cap \Omega'^0_{q_n^h} \\ 0, & \text{otherwise} \end{cases} .$$

Based on the random switching times and the discretization of the reset maps, $\forall (x, q) \in S$ such that $0 \leq a_q(x) \leq \gamma(h)$ the transition probabilities of the approximating MDP for states $(x, q), x \in O_q$ are defined as

$$\tilde{p}\left((x, q), (x', q')|u\right) = \begin{cases} (1 - e^{-\lambda_q(x)\Delta t^h(x,q,u)}) p((x, q), (x', q')|u), \\ \text{if } q \neq q' \text{ and } x' \in \Xi \\ e^{-\lambda_q(x)\Delta t^h(x,q,u)} p\left((x, q)(x', q')\right), \\ \text{if } q = q' \text{ and } = x \pm he_i \end{cases} .$$

By the construction of the switching rate function, as $h \to 0$, the cover $\{\Omega'_q\}$ converges to the original partition $\{\Omega_q\}$ and the approximating process preserves local consistency.

4.3 Boundary of the Target Set

We also define a random stopping rule when the state approaches the boundary of the target set $G \subset \Omega_{q_f}$. The process stops at step n with probability $1 - e^{-\lambda_G(\xi_n^h)\Delta t^h(\xi_n^h, q_n^h, u_n^h)}$ if $\xi_n^h \in N_h(\partial G) \cap \Omega^0_{q_f}$, and with probability 1 if $\xi_n^h \in \partial G$.

4.4 Reflective Boundaries

In practical applications, the physical process is usually constrained in a bounded state space. Reflective boundaries are introduced to approximate such constraints. For the approximating MDPs, the constraints are modeled as reflective (or constrained) boundaries equipped with reflection directions that point into the state space. The process is reflected back when it tries to violate the constraints. Local consistency can be satisfied at the reflective boundaries in a straightforward manner by computing the transition probabilities based on the reflection directions [19]. It is assumed that the transition probabilities do not depend on the controls and that the reflections occur instantaneously and hence, the corresponding interpolation intervals are zero.

5 Computational Methods for Optimal Control

We have described how to approximate the original stochastic hybrid system by a locally consistent MDP. For the optimal control problem, one also needs to approximate the original cost function by one which is appropriate for the MDP. Then, numerical algorithms can be used to compute an optimal value function and design a feedback controller for the system.

Consider the approximating MDP $\{\xi_n^h, q_n^h\}$ with transition probabilities $\tilde{p}((x,q),(y,q')|u)$ and denote ν_h the stopping time representing that ξ_n^h reaches the target set G. Then, assuming that the discounting is constant in the intervals $[t_n^h, t_{n+1}^h)$ the cost (2) can be approximated by

$$W^h(x_0, q_0, u) = E\left[\sum_{n=0}^{\nu_h} e^{-\beta t_n^h} c(\xi_n^h, q_n^h, u_n^h) + e^{-\beta t_{\nu_h}^h} g(\xi_{\nu_h}^h)\right]$$

where $c(\xi_n^h, q_n^h u_n^h) = k(\xi_n^h, q_n^h, u_n^h)\Delta\tau_n^h$.

Assuming that the above sum is well-defined and bounded, minimizing the cost is a discrete problem that can be solved using standard dynamic programming algorithms based on policy or value iteration methods. If $\beta = 0$ then it is required that the target set G is reachable from the initial state. Reachability is satisfied if for the initial state $\xi_0^h \in X \setminus G$ there exists a control sequence $\{u_n^h\}, n < \infty$ and a path $\{(\xi_0^h, q_0^h), (\xi_1^h, q_1^h), \ldots, (\xi_{\nu_h}^h, q_{\nu_h}^h)\}$ such that the

$$P(\xi_n^h \in G | \xi_0^h, q_0^h, u_0^h, \ldots, \xi_{\nu_h-1}^h, q_{\nu_h-1}^h, u_{\nu_h-1}^h) > 0.$$

Since the state space of the approximating MDP is assumed to be finite, reachability of the target set can be tested using the transition probability matrix.

We can define the optimal value function

$$V^h(x_0, q_0) = \inf_u W^h(x_0, q_0, u)$$

and using a standard dynamic programming argument, we can derive the equation

$$V^h(x,q) = \min_u \left[\sum_{y,q'} \tilde{p}((x,q),(y,q')|u) V^h(y,q') + c(x,q,u)\right]$$

if $x \in X \setminus G$ and $V^h(x,q) = g(x)$, if $x \in G$

For the reflective boundaries, we have considered that the transition probabilities are independent of the control and the reflections are instantaneous. Let the reflections be defined by a set of vectors of unit length $r_q(x)$, a cost $c_r^T(x)E[\Delta\xi_n^h]$ is associated with the reflective boundary such that $c_r^T(x) r_q(x) \geq 0$ and $c_r^T(x) \geq 0$ elementwise to approximate the cost of the unconstrained process. The equation for the optimal value for point on the reflective boundaries is

$$V^h(x,q) = \sum_{y,q'} \tilde{p}((x,q),(y,q')) V^h(y,q') + c_r^T(x) E[\Delta x].$$

The optimal value function can be computed by the value iteration

$$V_{n+1}^h(x,q) = min_u \left[\sum_{y,q'} \tilde{p}((x,q),(y,q')|u) V_n^h(y,q') + c(x,q,u) \right]$$

with the boundary conditions described above. The value function $V^h(x,q)$ is a discrete approximation of the optimal cost for the hybrid system. Finally, given the discrete optimal value function, a feedback control scheme can be designed for computing $u(x)$. For (x,q) with $x \in \Omega_q$, the control law is given by

$$u(x) = \arg\min_u \left\{ \frac{\partial V^h(x,q)}{\partial x} f(x,q,u) + k(x,q,u) \right\} \quad (5)$$

The gradient of $V^h(x,q)$ can be approximated as a weighted function of the differences ΔV^h of the values at the grid points. The advantage of the discretization method based on the locally consistent MDPs is that the cost of the approximating discrete-time process converges weakly to the original cost as shown in Section 6.

6 Convergence

Given the approximation parameter h, the SHS has been approximated by a locally consistent MDP. The original cost has been expressed in terms of the MDP and computational methods can be used to compute the optimal value function $V^h(x,q)$. In this section, we will show that $V^h(x,q)$ converges to the optimal value function $V(x,q)$ of the original problem. Convergence of the approximating processes for control diffusions has been shown in [19]. We extend this approach to address the difficulties that arise due to the hybrid dynamics.

6.1 Convergence of the Approximating Processes

Consider the locally consistent approximating process $\{\xi_n^h, q_n^h\}$ and the optimal control input $\{u_n^h\}$ and denote $\{t_i^h\}$ the sequence of switching times. First, a continuous time interpolation $\{\psi^h(t), q^h(t)\}$ is constructed so that $\{\psi^h(t)\}$ is a Markov process. This will allow the construction of the Wiener process $w(t)$ as $h \to 0$. Denote the moments of change of $\psi^h(t)$ by $\tau_n^h, n < \infty$ with $\tau_0^h = 0$. To ensure that $\psi^h(t)$ is a Markov process, the interpolation intervals $\Delta t^h(x,q,u)$ are considered not to be deterministic but they are described by an exponential distribution with mean $\Delta t^h(x,u)$, i.e.,

$$P[\Delta\tau_n^h \le t | \xi_n^h = x, q_n^h = q, u_n^h = u] = 1 - e^{-\frac{t}{\Delta t^h(x,q,u)}}$$

$$E[\Delta\tau_n^h | \xi_n^h = x, q_n^h = q, u_n^h = u] = \Delta t^h(x,q,u).$$

Using these new intervals we define

$$\psi^h(\tau_n^h) = \xi_n^h, n < \infty \tag{6}$$

$$\psi^h(t) = \sum_{i:\tau_{i+1}^h \le t} \Delta\xi_i^h + \xi_0^h \tag{7}$$

and

$$q^h(t) = q_n^h, t \in [\tau_n^h, \tau_{n+1}^h)$$

$$u^h(t) = u_n^h, t \in [\tau_n^h, \tau_{n+1}^h).$$

From (7) we can write

$$\begin{aligned} \psi^h(t) &= \xi_0^h + \sum_{i:\tau_{i+1}^h \le t} \Big[E[\Delta\xi_n^h] + \Delta\xi_n^h] - E[\Delta\xi_n^h]\Big] \\ &= \xi_0^h + \sum_{i:\tau_{i+1}^h \le t} E[\Delta\xi_n^h] + B_n^h \end{aligned}$$

where $B_n^h = \sum_{i:\tau_{i+1}^h \le t}[\Delta\xi_n^h] - E[\Delta\xi_n^h]]$ is an $\mathbb{R}^d$-valued discrete martingale. Denote $\Delta t_i^h = \Delta t^h(\xi_i^h, q_i^h, u_i^h)$, then by local consistency $\sum_{i:\tau_{i+1}^h \le t} E[\Delta\xi_n^h] = \sum_{i:\tau_{i+1}^h \le t} f(\xi_n^h, q_i^h, u_i^h)\Delta t_i^h + o(\Delta t_i^h)$ and B_n^h has quadratic variation $\sum_{i:\tau_{i+1}^h \le t} a(\xi)\Delta t_i^h + o(\Delta t_i^h)$. As $h \to 0$, we get

$$\psi^h(t) = \psi^h(0) + \int_{t_0}^t f(\psi^h(s), q^h(s), u^h(s))ds + \delta_1^h(t) + B^h(t) \tag{8}$$

where

$$B^h(t) = \int_{t_0}^t a(\psi^h(s))ds + \delta_2^h(t)$$

with $E[\sup_{s\le t} \delta_1^h(s)] \to 0$ and $E[\sup_{s\le t} \delta_2^h(s)] \to 0$.

The computational methods will give an optimal control sequence $\{u_n^h\}$. The optimal control may not exist as $h \to 0$. To show convergence, a relaxed control representation [25] is employed. Denote the space of the relaxed control as $\mathcal{A}$ and $\mathcal{B}_{\mathcal{A}}$ and $\mathcal{B}_{\mathcal{A}\times[0,\infty)}$ the Borel σ-algebras on $\mathcal{A}$ and $\mathcal{A} \times [0, \infty)$ respectively. A relaxed control representation can be obtained by defining probability measures μ_t on $\mathcal{B}_{\mathcal{A}}$ and μ on $\mathcal{B}_{\mathcal{A}\times[0,\infty)}$ as

$$\begin{aligned} \mu_t(A) &= I_A(\alpha(t)) \\ \mu(A \times [0, t]) &= \int_0^t \mu_t(A)ds \end{aligned}$$

where $\alpha(t) \in A \subseteq \mathcal{A}$ and I_A is the characteristic function for the set A.

Denote μ_t^h and μ^h the corresponding probability measures for the sequence $\psi^h(t)$, then (8) can be written as

$$\begin{aligned} \psi^h(t) = \psi^h(0) + \\ \int_{t_0}^t \int_{\mathcal{A}} f(\psi^h(s), q^h(s), \alpha^h(s))\mu_s^h(d\alpha)ds + \delta_1^h(t) + B^h(t) \end{aligned} \tag{9}$$

In the following, we prove that $\{\psi^h(t), q^h(t)\}$ converges weakly to the execution of the SHS. Let E denote a metric space and $D_E[0,\infty)$ the set of functions that are continuous from the right and have limits from the left. The $\psi^h(t)$ and $q^h(t)$ are viewed as elements of $D_E[0,\infty)$ for $E = \mathbb{R}^d$ and $\mathbb{R}$ respectively. The difference with the results of [19] is that we show convergence of the switching times and then assuming finitely many switchings in a bounded interval we show convergence for the hybrid process based on the weak convergence results for $D_E[0,\infty)$ in [8] (Thm 7.8).

Theorem 3 *Consider the locally consistent approximating process $\{\xi_n^h, q_n^h\}$ with an admissible control sequence $\{u_n^h\}$ and a sequence of (random) switching times $\{t_i^h\}$. Let $\{\psi^h(t), q^h(t)\}$ be the continuous time Markov interpolation and $\mu^h(\cdot)$ a relaxed control representation of $\{u_n^h\}$ for $\psi^h(t)$. Then $\{\psi^h(t), q^h(t)\}$ converges weakly to the execution of the SHS.*

Proof To prove convergence of the sequence of switching times, without loss of generality we will consider only t_1^h. Since $t_1^h \in [0,\infty)$ the range of t_1^h is compact and therefore the sequence (as indexed by h) is tight. Hence, t_1^h is relatively compact and contains a weakly convergent subsequence with limit denoted by $\bar{t}_1$. Using a similar argument, $\mu^h(\cdot)$ is tight and has a weakly convergent subsequence with limit $\mu(\cdot)$. From the results in [19], the martingale $B^h(t)$ can be written as

$$B^h(t) = \int_{t_0}^{t} \sigma(\psi^h(s))dw^h(s) + \epsilon_1^h(t)$$

where $E[\sup_{s\leq t} |\epsilon_1^h(t)| \to 0$. Further, $w^h(t)$ is tight and has a limit $w(t)$ which is a Wiener process. By the assumption of finitely many switchings in a bounded interval $E[q^h(\nu_h+\delta) - q^h(\nu_h)] \to 0$ for any stopping time ν_h. Therefore $q^h(t)$ is tight and we denote $q(t)$ its limit. Finally, from the above results, the boundness of f, and the properties of the stochastic integral in (10) $\psi^h(t)$ is tight and we denote the limit by $x(t)$. Consider the mapping $\hat{t}_1 : D[0,\infty) \to [0,\infty]$ given by $\hat{t}_1(\phi(t)) = \inf(t > t_0 : \ \phi(t) \notin \Omega^0_{q_0})$. Then by the randomized switching rule, $\hat{t}_1$ is continuous with probability 1 and

$$\bar{t}_1 = \lim_h t_1^h = \lim_h (\hat{t}_1(\psi^h(t)) = \hat{t}_1(x(t) = t_1.$$

By the definition of $q^h(t)$ and since $\bar{t}_1 = t_1$, $q(t)$ is piecewise constant. By the assumption of finitely many switchings in a bounded interval, $\psi^h(t_i^h) \to x(t_i)$ ([8], Thm 7.8) and therefore, the integral in (10) can be written as

$$\int_{t_0}^{t}\int_{\mathcal{A}} f(\psi^h(t), q^h(t), \alpha^h(s))\mu_s^h(d\alpha)ds \to \int_{t_0}^{t}\int_{\mathcal{A}} f(x(t), q(t), \alpha(s))\mu_s^h(d\alpha)ds$$

Finally, $B^h(t)$ converges to $\int_{t_0} \sigma(x(s))dw(s)$ [19]. □

6.2 Convergence of the Optimal Cost

The next theorem shows convergence of the optimal cost. The proof follows from Theorem 3 and the results in [19]. The cost for the continuous time interpolation $\{\psi^h(t), q^h(t)\}$ with control sequence $\mu^h(t)$ is

$$W^h(x_0, q_0, \mu^h) =$$
$$E\left[\int_{t_0}^{\tau^h} e^{-\beta s} k(\psi^h(s), q^h(s), \alpha)\mu_s^h(d\alpha)ds + e^{-\beta\tau^h} g(\psi^h(\tau_h))\right].$$

Denote $V'^h(x, q) = \inf_{\mu^h} W^h(x, q, \mu^h)$ and let $V^h(x, q)$ denote the optimal value function for the process $\{\xi_n^h, q_n^h\}$ with an admissible control sequence $\{u_n^h\}$. Then $|V'^h(x, q) - V^h(x, q)| \to 0$ as $h \to 0$ and by abuse of notation we will use $V(h, q)$ to denote the optimal value function for both processes.

Theorem 4 *Consider the locally consistent approximating process* $\{\xi_n^h, q_n^h\}$ *with an admissible control sequence* $\{u_n^h\}$ *and a sequence of (random) switching times* $\{t_i^h\}$*. Let* $\{\psi^h(t), q^h(t)\}$ *be the continuous time Markov interpolation and* $\mu^h(\cdot)$ *a relaxed control representation of* $\{u_n^h\}$ *for* $\psi^h(t)$*. If* $\beta > 0$ *or if* $\beta = 0$ *and the sequence* τ_h *is uniformly integrable then* $W^h(x_0, q_0, \mu^h) \to W(x_0, q_0, \mu)$ *and* $V^h(x, q) \to V(x, q)$*.*

Proof The assumption that either $\beta > 0$ or $\beta = 0$ and the sequence τ_h is uniformly integrable guarantees boundness of the cost. Then by the continuity of the exit times we get $W^h(x_0, q_0, \mu^h) \to W(x_0, q_0, \mu)$. Since $\liminf_h \tau^h \to \tau$, by the weak convergence and Fatou's lemma [8] we get $\liminf_h W^h(x_0, q_0, \mu^h) \geq W(x_0, q_0, \mu)$ and $\liminf_h V^h(x, q) \geq V(x, q)$.

The relaxed control can be approximated by a piecewise constant control $u^\epsilon(t)$ with relaxed control representation $\mu^\epsilon(\cdot)$ which is ϵ-optimal and results in a solution $(x^\epsilon(t), q^\epsilon(t)$ such that $|W(x_0, q_0, \mu^\epsilon) - W(x_0, q_0, \mu)| \leq \epsilon$.

Because the moments of change of $\psi^h(t)$ are not deterministic, they may not coincide with the the times the control changes in the discrete time approximation. By slightly changing the piecewise constant control it can be shown that the approximating process $(\psi^h(t), q^h(t))$ with control $\mu^h(\cdot)$ converges weakly to the solution $(x^\epsilon(t), q^\epsilon(t))$ with control $\mu^\epsilon(\cdot)$ and therefore $V^h(x, q) \leq W^h(x, q, \mu^h) \to W(x, q, \mu^\epsilon)$ [19]. Then ϵ-optimality implies that $W(x, q, \mu^\epsilon) + \delta(\epsilon) \leq V(x, q) + \delta(\epsilon) + \epsilon$ where $\delta\epsilon \to 0$ as $\epsilon \to 0$ and therefore $\limsup_h V^h(x, q) \leq V(x, q)$. □

7 Example

We illustrate the proposed approach using a simplified model of a truck with flexible transmission presented in [13]. The system is described by

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= -x_2 + x_3 \\ dx_3 &= -x_2 + g_q(x_2) u dt + \sigma dw, \\ q &= 1, 2,\ -0.1 \leq u \leq 1.1\ \sigma = 0.01 \end{aligned}$$

where x_1, x_2 and x_3 are the position, velocity, and the rotational displacement of its transmission shaft respectively. The efficiency for gear q is $g_q(x)$ shown in Figure 2(a), u is the throttle, and dw is a scalar Wiener process. We have modified the model of [13] by assuming that gears switches occur at the speed of equal efficiency between the gears ($x_2 = 0.5$) and therefore, the switching boundary is defined by $A = \{x : x_2 = 0.5\}$.

The objective is to drive the state (x_0, q_0) to the target set

$$G = \{x \in \Re^2 : \frac{1}{2} x^T x \leq 0.25\}$$

while minimizing the cost

$$W(x_0, q_0, u) = E\left[\int_0^\tau k(x(s), q(s), u(s)) ds + g(x(\tau))\right]$$

where $k(x, q, u) = 1$ and $g(x) = \frac{1}{2} x^T x$. First, we approximate the system by an MDP over the region

$$X = \{x : -5 \leq x_1 \leq 1, 0 \leq x_2 \leq 1.5, -0.5 \leq x_3 \leq 1.5\}$$

using a uniform grid with approximation parameter $h = 0.25$.

The reflective boundary is defined as an outer approximation of X by expanding by h in all directions. For the corner points, we select the reflection direction $r(x)$ as the vector of length h in the direction of the diagonal and we assume that the transition probabilities are independent of the control u. For the remaining points on the reflective boundary, we select $r(x)$ as the normal vector of length h pointing inside X. We transform the partition of the state space to a cover by defining two new boundaries $A = \{x : x_2 = 0.5 \pm 2h\}$ and the switching rate functions by $\lambda(x) = \frac{0.5}{\ln 0.5} \ln(1 \mp \frac{x_2 - 0.5}{2h})$ respectively. Everytime a switching occurs, we reset the continous state to guarantee finitely many switchings. We define the transition probabilities for local consistency as defined in Section 4. The optimal value function $V(x)$ is computed using an iteration method in value space. The results shown in Figure 2 are obtained by simulating the SHS model in continuous-time (using Simulink) where the control law is computed by (5) using multilinear interpolation. Except the stochastic nature of the state trajectory, the results are very similar to those presented in [13]. The advantage

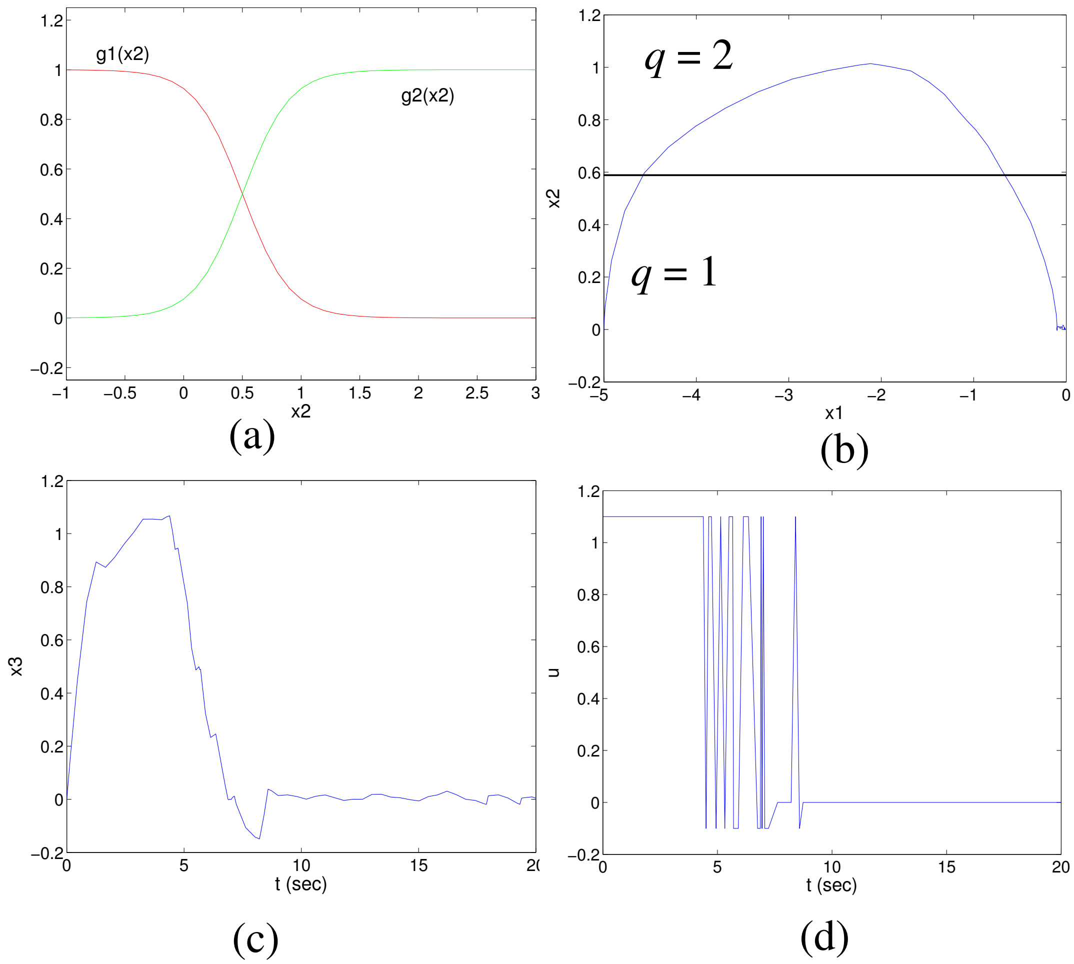

Figure 2: (a) Gear profiles, (b),(c), and (d) Simulation results

of the approach is that this solution which is based on a discrete approximation that preserves the local mean and variance of the original system.

8 Conclusions and Future Work

The paper employs an approximation method for solving the optimal control problem for stochastic hybrid systems based on locally consistent Markov decision processes that preserve the local mean and covariance of the original system. The approach gives rise to several significant problems. A fundamental challenge is to identify and characterize problems that can be solved based on the approximations whose solutions converge to their correct values as the approximation parameter goes to zero. Another challenge is to develop scalable numerical methods that can be applied to large systems. Finally, the approach can be extended to more general stochastic hybrid systems that may include both continuous and discrete controls and state jumps.

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